

When Lexicographic Product of Two po-Groups has the Riesz Decomposition Property

Anatolij Dvurečenskij^{1,2}, Omid Zahiri³

¹*Mathematical Institute, Slovak Academy of Sciences,*

Štefánikova 49, SK-814 73 Bratislava, Slovakia

²*Depart. Algebra Geom., Palacký Univer., 17. listopadu 12,*

CZ-771 46 Olomouc, Czech Republic

³*University of Applied Science and Technology, Enghelab Av., Tehran, Iran*

dvurecen@mat.savba.sk zahiri@protonmail.com

Abstract

We study conditions when a certain type of the Riesz Decomposition Property (RDP for short) holds in the lexicographic product of two po-groups. Defining two important properties of po-groups, we extend known situations showing that the lexicographic product satisfies RDP or even RDP_1 , a stronger type of RDP. We recall that a very strong type of RDP, RDP_2 , entails that the group is lattice ordered. RDP's of the lexicographic products are important for the study of lexicographic pseudo effect algebras, or perfect types of pseudo MV-algebras and pseudo effect algebras, where infinitesimal elements play an important role both for algebras as well as for the first order logic of valid but not provable formulas.

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1 Introduction

In the last decades we observe that there is a growing interest to the study of some algebraic structures using lattice ordered groups or po-groups both for Abelian and non-Abelian ones. A prototypical situation is due to Mundici, see [Mun, CDM], when any MV-algebra is represented as an interval in a unital Abelian ℓ -group. This result was extended in [Dvu1] where there was proved that pseudo MV-algebras, a non-commutative generalization of MV-algebras, see [GeIo, Rac], can be represented by intervals in unital ℓ -groups not necessarily Abelian.

For mathematical foundations of quantum mechanics, Foulis and Bennett introduced in [FoBe] effect algebras which are partial algebras with a partially defined operation $+$, where $a + b$ means disjunction of two mutually excluded events a and b . These effect algebras are in many cases also intervals in Abelian po-groups (= partially ordered groups). A sufficient condition for such a po-group representation is the Riesz Decomposition property, RDP, of the effect algebra and of the po-group, as it follows from [Rav]. RDP means roughly speaking a possibility to perform a joint refinement of any two decompositions of the same element, and po-groups with RDP are intensively studied in literature, see e.g. [Fuc2, Go]. Recently effect algebras have been extended to non-commutative algebras, called pseudo effect algebras in [DvVe1, DvVe2]. Also if such a pseudo effect algebra satisfies a stronger form of RDP, namely RDP_1 , then the pseudo effect algebra is an interval in a po-group with RDP_1 not necessarily Abelian, see

[DvVe1, DvVe2]. If we define yet a more stronger type of RDP, RDP_2 , then the corresponding pseudo effect algebra is even a pseudo MV-algebra.

A perfect MV-algebra is an MV-algebra where each element is either an infinitesimal or a co-infinitesimal. Di Nola and Lettieri [DiLe1, DiLe2] showed that such MV-algebras can be represented as an interval in the lexicographic product $\mathbb{Z} \overrightarrow{\times} G$ with strong unit $(1, 0)$ for some Abelian ℓ -group G . Any perfect effect algebra with RDP was represented also as an interval in the lexicographic product $\mathbb{Z} \overrightarrow{\times} G$, where G is an Abelian po-group with RDP, [Dvu2].

We note that perfect MV-algebras have no parallels in the realm of Boolean algebras because perfect MV-algebras are not semisimple. The logic of perfect MV-algebras has an analogue in the Lindenbaum algebra of the first order Lukasiewicz logic which is not semisimple, because the valid but unprovable formulas are precisely the formulas that correspond to co-infinitesimal elements of the Lindenbaum algebra, see e.g. [DiGr].

A more general type of MV-algebras which are intervals in the lexicographic products $A \overrightarrow{\times} G$, where A is a linearly ordered group and G is an ℓ -group, were described in [DFL]. Lexicographic types of pseudo MV-algebras were studied in [Dvu3], and lexicographic types of effect algebras were presented in [Dvu4] as an interval in $A \overrightarrow{\times} G$, where A is a linearly ordered group and G is a po-group with RDP. As we see, for lexicographic pseudo MV-algebras or pseudo effect algebras, the lexicographic product of two po-groups with RDP play a crucial role, see also [DvKo, Dvu3], where it was shown that if A is an antilattice po-group with RDP and G a directed po-group with RDP, then $A \overrightarrow{\times} G$ has RDP.

Therefore, the study of perfect MV-algebras, perfect pseudo MV-algebras or perfect pseudo effect algebras is tightly connected with an important phenomenon of the first order Lukasiewicz logic on one side and on the lexicographic product of two po-groups with some kind of RDP on the second side. This is for us a good excuse to study lexicographic product of po-groups.

Thus our main goal of the present paper is to extend situations when the lexicographic product $A \overrightarrow{\times} G$ satisfies RDP.

The paper is organized as follows. The second section is an introduction to theory of po-groups. Here we define some kinds of RDP's, we show their basic properties as well as a correct example that $\text{RDP} \neq \text{RDP}_1$, which besides is of the form of the lexicographic product. The third section describes situations when $A \overrightarrow{\times} G$ satisfies RDP, if A is not necessarily an antilattice po-group with RDP. In the fourth section we define the com-directness property of a directed po-group G which is stronger than directness and for Abelian po-groups they are equivalent. We show that then $A \overrightarrow{\times} G$ will have RDP for each A and G with RDP. The fifth section is a continuation of the research. We define Non-Comparability Directness Property and we show its importance for the lexicographic product. In Conclusion we summarize our results and indicate possible applications for lexicographic pseudo effect algebras and some open questions are presented.

2 Riesz Decomposition Properties of po-Groups

We remind that a *po-group* is an additively written group $(G; +, 0)$ endowed with a partial order \leq such that $g \leq h$ implies $a + g + b \leq a + h + b$ for all $a, b, h, g \in G$. The *lexicographic product* of two po-groups $(G_1; +, 0)$ and $(G_2; +, 0)$ is the direct product $G_1 \times G_2$ endowed with the lexicographic ordering \leq such that $(g_1, h_1) \leq (g_2, h_2)$ iff $g_1 < g_2$ or $g_1 = g_2$ and $h_1 \leq h_2$ for $(g_1, h_1), (g_2, h_2) \in G_1 \times G_2$. If the order \leq implies that G is a lattice, we say that G is a *lattice ordered po-group*, or simply an ℓ -group.

We denote by $G^+ := \{g \in G \mid g \geq 0\}$ and $G^- = \{g \in G \mid g \leq 0\}$. An element $u \in G^+$ is said to be a *strong unit* (or order unit) if, given $g \in G$, there is an integer $n \geq 0$ such that $g \leq nu$. A *unital po-group* is a pair (G, u) , where G is a po-group and u is a fixed strong unit of G . If (H, u) is a unital po-group, then $(H \overrightarrow{\times} G, (u, 0))$ is a unital po-group.

For more information about po-groups we recommend for example the following books [Dar, Fuc1, Gla].

The *center* of a group G is the set $Z(G) = \{x \in G \mid x + y = y + x \text{ for all } y \in G\}$.

A po-group G is *directed* if, given $g_1, g_2 \in G$, there is $h \in G$ such that $g_1, g_2 \leq h$. This is equivalent to the property: Given $g_1, g_2 \in G$, there is $h \in G$ such that $g_1, g_2 \geq h$; or equivalently $G^+ - G^+ = G$.

A poset $(P; \leq)$ is said to be an *antilattice* if only comparable elements $a, b \in P$ have a joint or meet in P . A directed po-group G is an antilattice iff $a \wedge b = 0$ implies $a = 0$ or $b = 0$. For example, antilattices are (i) every linearly ordered group, (ii) $\mathcal{B}(H)$, the group of Hermitian operators of a Hilbert space H , [LuZa, Thm 58.4], (iii) $G = \mathbb{R}^2$ with the positive cone of all (x, y) such that either $x = y = 0$ or $x > 0$ and $y > 0$; in addition, G is an antilattice with RDP, but G is not a lattice.

In the literature, see e.g. [Fuc2, Go, DvVe1], there is a whole variety of the Riesz Decomposition Properties.

We say that a po-group $(G; +, 0)$ satisfies

- (i) the *Riesz Interpolation Property* (RIP for short) if, for $a_1, a_2, b_1, b_2 \in G$, $a_1, a_2 \leq b_1, b_2$ implies there exists an element $c \in G$ such that $a_1, a_2 \leq c \leq b_1, b_2$;
- (ii) RDP_0 if, for $a, b, c \in G^+$, $a \leq b + c$, there exist $b_1, c_1 \in G^+$, such that $b_1 \leq b$, $c_1 \leq c$ and $a = b_1 + c_1$;
- (iii) RDP if, for all $a_1, a_2, b_1, b_2 \in G^+$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$;
- (iv) RDP_1 if, for all $a_1, a_2, b_1, b_2 \in G^+$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$, and $0 \leq x \leq c_{12}$ and $0 \leq y \leq c_{21}$ imply $x + y = y + x$;
- (v) RDP_2 if, for all $a_1, a_2, b_1, b_2 \in G^+$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in G^+$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$, and $c_{12} \wedge c_{21} = 0$.

If, for $a, b \in G^+$, we have for all $0 \leq x \leq a$ and $0 \leq y \leq b$, $x + y = y + x$, we denote this property by $a \text{ com } b$.

The RDP will be denoted by the following table:

a_1	c_{11}	c_{12}
a_2	c_{21}	c_{22}
	b_1	b_2

For Abelian po-groups, RDP , RDP_1 , RDP_0 and RIP are equivalent.

By [DvVe1, Prop 4.2], for directed po-groups we have

$$\text{RDP}_2 \Rightarrow \text{RDP}_1 \Rightarrow \text{RDP} \Rightarrow \text{RDP}_0 \Leftrightarrow \text{RIP}, \quad (2.1)$$

but the converse implications do not hold, in general. More precisely, in [DvVe1, Prop 4.2(ii)], there was proved (i) a directed po-group G satisfies RDP_2 iff G is an ℓ -group, and in general, (ii) $\text{RDP}_1 \not\Rightarrow \text{RDP}_2$, (iii) $\text{RDP}_0 \not\Rightarrow \text{RDP}$.

Remark 2.1. In [DvVe1, Prop 4.2(ii)], there was found an example of a po-group with RDP but RDP_1 fails in it. Unfortunately, as we now show, [DvVe1, Ex 3.5] has a gap because, the po-group from that example is in fact Abelian, so it satisfies both RDP as well as RDP_1 .

Proof. Let G be an additive group generated by the countably many elements g_0, g_1, \dots , let $v : (G; +, 0) \rightarrow (\mathbb{R}; +, 0)$ be the homomorphism determined by the conditions $v(g_i) = (1/2)^i$, $i = 0, 1, \dots$, and let G fulfil the condition that every $a \in G$ such that $v(a) = 0$ commutes with each element $b \in G$. Define a partial order in G by setting $G^+ := \{x \in G \mid x = 0 \text{ or } v(x) > 0\}$. This means that we have for $a, b \in G$ $a \leq b$ iff $a = b$ or $v(a) < v(b)$. In [DvVe1, Ex 3.5], there was proved that G satisfies RDP .

In what follows, we show that every g_i commutes with each g_j for $i, j = 0, 1, \dots$. Indeed, let $i < j$. For the element $g_i - 2^{j-i}g_j$, we have $v(g_i - 2^{j-i}g_j) = 0$, so that it commutes with every element of G , in particular with g_j . Then $g_j + (g_i - 2^{j-i}g_j) = (g_i - 2^{j-i}g_j) + g_j = g_i + g_j - 2^{j-i}g_j$ so that $g_j + g_i = g_j + g_i$. Since the set $\{g_0, g_1, \dots\}$ generates G , G is Abelian. Whence, it satisfies RDP_1 , which contradicts the statement in [DvVe1, Ex 3.5]. \square

The following example is a simple one showing how we can create non-Abelian po-groups $H \vec{\times} G$ with RDP or RDP₁.

Example 2.2. Let $(G; +, 0)$ be a po-group with RDP (RDP₁) and $(A; +, 0)$ be a non-Abelian group. Consider the trivial ordering on A . Then $A \vec{\times} G$ is a non-Abelian po-group. It can be easily seen that $A \vec{\times} G$ has RDP (RDP₁). Indeed, for each positive element $(a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2) \in A \vec{\times} G$, such that $(a_1, a_2) + (b_1, b_2) = (c_1, c_2) + (d_1, d_2)$ we have $a_1 = b_1 = c_1 = d_1 = 0$ and so we can find an RDP table (RDP₁ table) for it.

In particular, if G is in addition with RDP, then $A \vec{\times} G$ is a non-Abelian po-group satisfying RDP₁.

The following example shows that there exists a po-group such that $\text{RDP} \not\Rightarrow \text{RDP}_1$. This example corrects the implication in the example of [DvVe1, Prop 4.2(ii), Ex. 3.5], see Remark 2.1.

First we present the following definition.

Let $\{A_i \mid i \in I\}$ be a system of po-groups. The direct product $\prod_{i \in I} A_i$ consists of elements of the form $(a^i)_{i \in I}$ or simply (a^i) , where $a^i \in A_i$ for all $i \in I$. Besides the product ordering \leq on the direct product $\prod_{i \in I} A_i$, defined by $(a^i)_{i \in I} \leq (b^i)_{i \in I}$ iff $a^i \leq b^i$ for all $i \in I$, we define the *strict product ordering*, \leq , defined by $(a^i)_{i \in I} \leq (b^i)_{i \in I}$ iff either $a^i = b^i$ for all $i \in I$ or $a^i < b^i$ for all $i \in I$.

Lemma 2.3. Consider the po-group $\mathbb{R} \times \mathbb{R}$ with the strict product ordering \leq . Let $(G; +, 0)$ be a non-Abelian directed po-group with RDP. Then $(\mathbb{R} \times \mathbb{R}) \vec{\times} G$ satisfies RDP. If G satisfies RDP₁, then $((\mathbb{R} \times \mathbb{R}) \vec{\times} G, ((1, 1), 0))$ is a unital po-group satisfying RDP, but RDP₁ fails for it.

Proof. Let $((a_1, a_2), x), ((b_1, b_2), y), ((c_1, c_2), z), ((d_1, d_2), u) \geq 0$ in $(\mathbb{R} \times \mathbb{R}) \vec{\times} G$ be such that $((a_1, a_2), x) + ((b_1, b_2), y) = ((c_1, c_2), z) + ((d_1, d_2), u)$. Then either $a_1, a_2 > 0$ or $a_1, a_2 = 0$, and similarly for the elements $(b_1, b_2), (c_1, c_2), (d_1, d_2)$.

Inasmuch as \mathbb{R} is a linearly ordered group, from [Dvu3, Thm 3.1], we conclude that, for $(a_1, x) + (b_1, y) = (c_1, z) + (d_1, u)$, there is an RDP decomposition

$$\begin{array}{c|cc} (a_1, x) & (n'_{11}, c_{11}) & (n'_{12}, c_{12}) \\ (b_1, y) & (n'_{21}, c_{21}) & (n'_{22}, c_{22}) \\ \hline & (c_i, z) & (d_i, u) \end{array}$$

and an RDP decomposition for $a_2 + b_2 = c_2 + d_2$

$$\begin{array}{c|cc} a_2 & n''_{11} & n''_{12} \\ b_2 & n''_{21} & n''_{22} \\ \hline & c_2 & d_2 \end{array}.$$

Put

$$\begin{array}{c|cc} ((a_1, a_2), x) & ((n'_{11}, n''_{11}), c_{11}) & ((n'_{12}, n''_{12}), c_{12}) \\ ((b_1, b_2), y) & ((n'_{21}, n''_{21}), c_{21}) & ((n'_{22}, n''_{22}), c_{22}) \\ \hline & ((c_1, c_2), z) & ((d_1, d_2), u) \end{array} \cdot (A)$$

If, one pair of the elements, e.g. $(a_1, a_2) = (0, 0)$, then $(n'_{11}, n''_{11}) = (0, 0) = (n'_{12}, n''_{12})$, and for this case, (A) gives an RDP table.

Therefore, we can assume that $a_1, a_2, b_1, b_2, c_1, d_2 > 0$. We claim that in the table (A) we can assume that all $n'_{ij}, n''_{ij} > 0$ for $i, j = 1, 2$. Indeed, take e.g. the table

$$\begin{array}{c|cc} a_1 & n'_{11} & n'_{12} \\ b_i & n'_{21} & n'_{22} \\ \hline & c_i & d_i \end{array}.$$

If, say $n'_{11} = 0$, then $n'_{12} > 0$ and $n'_{21} > 0$. Since n'_{12} and n'_{21} are comparable, there is $n_0 \in A_i$ such that $0 < n'_0 < n'_{12}, n'_{21}$. Then in the table

$$\begin{array}{c|cc} a_1 & n'_{11} + n'_0 & n'_{12} - n'_0 \\ b_1 & n'_{21} - n'_0 & n'_{22} + n'_0 \\ \hline & c_1 & d_1 \end{array}$$

all entries are strictly positive. In the same way we proceed with n''_{ij} .

Now we show that if G satisfies RDP_1 , then it does not hold in $(\mathbb{R} \times \mathbb{R}) \overrightarrow{\times} G$. Put $A = \mathbb{R} \times \mathbb{R}$ and let $\pi_1 : A \overrightarrow{\times} G \rightarrow A$ be the canonical projection map. Since G is non-Abelian, there are $x, y \in G$ such that $x + y \neq y + x$. Put $a, b \in G$ such that $x + y = a + b$. Consider the equation $((1, 4), x) + ((3, 7), y) = ((2, 3), a) + ((2, 8), b)$. For each RDP table

$$\begin{array}{c|cc} ((1, 4), x) & c_{11} & c_{12} \\ ((3, 7), y) & c_{21} & c_{22} \\ \hline & ((2, 3), a) & ((2, 8), b) \end{array},$$

we have $(0, 0) \leq \pi_1(c_{12})$ and $(0, 0) \leq \pi_1(c_{21})$. Since $(1, 4)$ and $(2, 3)$ are not comparable as well as $(3, 7)$ and $(2, 3)$, then $\pi_1(c_{12})$ and $\pi_1(c_{21})$ are non-zero elements of A and so $\pi_1(c_{12})$ and $\pi_1(c_{21})$ are strictly positive. We can select a strictly positive element (s, t) of A such that $(s, t) \leq \pi_1(c_{12})$, $(s, t) \leq \pi_1(c_{21})$ and $(s, t) \neq \pi_1(c_{12})$, $(s, t) \neq \pi_1(c_{21})$. Clearly, $((s, t), x) \leq c_{12}$ and $((s, t), y) \leq c_{21}$ and $((s, t), y) + ((s, t), x) \neq ((s, t), x) + ((s, t), y)$. \square

We note that for Abelian po-groups the equivalence RIP , RDP_0 and RDP was established in [Go, Prop 2.1] without assumption that G is directed. In [DvVe1, Prop 4.2], the implications $\text{RDP} \Rightarrow \text{RDP}_0 \Leftrightarrow \text{RIP}$ was proved for all po-groups under the assumption G is directed. In what follows, we prove that RDP_0 is equivalent to RIP for any po-group G not assuming G is directed.

Lemma 2.4. *In any po-group G , RDP_0 is equivalent to RIP .*

Proof. Let G satisfy RDP_0 and let $a_1, a_2 \leq b_1, b_2$. Then $b_i - a_j \geq 0$ for $i, j = 1, 2$, and

$$b_2 - a_1 = (b_2 - a_2) + (a_2 - b_1) + (b_1 - a_1) \leq (b_2 - a_2) + (b_1 - a_1).$$

Due to RDP_0 , there are $c_1, c_2 \in G^+$ such that $b_2 - a_1 = c_1 + c_2$ and $c_1 \leq b_2 - a_2$, $c_2 \leq b_1 - a_1$. If we put $c = c_2 + a_1$, we have $b_2 = c_1 + c_2 + a_1 = c_1 + c$ which entails $c \leq b_2$ and $a_1 \leq c$. On the other hand, $c - a_1 = c_2 \leq b_1 - a_1$ which gives $c \leq b_1$. Finally, $b_2 = c_1 + c \leq (b_2 - a_2) + c$, so that $a_2 \leq c$. Hence, G satisfies RIP .

The converse implication follows from [DvVe1, Prop 4.2]. \square

According to the latter result, the assumption of directness of a po-group G is superfluous in (2.1).

Proposition 2.5. *Let A be an antilattice po-group satisfying RDP (RDP_1). If for $a_1, a_2, b_1, b_2 \in A^+$ we have $a_1 + a_2 = b_1 + b_2$, where $a_1 \parallel b_1$, then there is an RDP (RDP_1) decomposition (n_{ij}) , $i, j = 1, 2$, in A such $n_{12}, n_{21} > 0$. In addition, in such a case, $n_{12}, n_{21} > 0$ and $n_{12} \parallel n_{21}$.*

Moreover, if (m_{ij}) , $i, j = 1, 2$, is an arbitrary RDP decomposition for $a_1 + a_2 = b_1 + b_2$, then $m_{12} > 0$ and $m_{21} > 0$.

Proof. Since a_1 and b_1 are not comparable, so are a_2 and b_2 , and hence $a_1, b_1 > 0$ as well as $a_2, b_2 > 0$. We assert that there is $n_0 \in A$ such that $0 < n_0 < a_1, b_1$. Suppose the converse. We show that then $a_1 \wedge b_1$ exists in A and $a_1 \wedge b_1 = 0$. Let $c \leq a_1, b_1$. Since RDP entails the Riesz Interpolation Property, Lemma 2.4, there is $d \in A$ such that $c, 0 \leq d \leq a_1, b_1$. Due to the assumptions, $d < a_1, b_1$ and $d = 0$, so that $c \leq 0$, and $a_1 \wedge b_1 = 0$ which is impossible because a_1 and b_1 are incomparable which proves the assertion.

Similarly, there is $m_0 \in A$ such that $0 < m_0 < a_2, b_2$. Hence, we have $(-n_0 + a_1) + (a_2 - m_0) = (-n_0 + b_1) + (b_2 - m_0)$, where all the elements in brackets are strictly positive. Due to RDP (RDP_1) of A , we have an RDP (RDP_1) table for $(-n_0 + a_1) + (a_2 - m_0) = (-n_0 + b_1) + (b_2 - m_0)$ as follows

$$\begin{array}{c|cc} -n_0 + a_1 & n_{11} & n_{12} \\ a_2 - m_0 & n_{21} & n_{22} \\ \hline & -n_0 + b_1 & b_2 - m_0 \end{array},$$

which gives

$$\begin{array}{c|cc} a_1 & n_0 + n_{11} & n_{12} \\ a_2 & n_{21} & n_{22} + m_0 \\ \hline & b_1 & b_2 \end{array},$$

where the elements in the upper left-side corner and in the lower right-side corner are strictly positive.

In other words, if $a_1 \parallel b_1$, there is always an RDP (RDP₁) table for $a_1 + a_2 = b_1 + b_2$

$$\begin{array}{c|cc} a_1 & n_{11} & n_{12} \\ a_2 & n_{21} & n_{22} \\ \hline & b_1 & b_2 \end{array}$$

such that $n_{11}, n_{22} > 0$.

Assume that for our RDP table $n_{12} = 0$. Then $n_{11} = a_1 \leq b_1$ which is impossible. In the similar way, we can prove that $n_{21} > 0$.

Now let n_{12} and n_{21} be comparable. Due to the equality $n_{11} + n_{12} + n_{21} + n_{22} = n_{11} + n_{21} + n_{12} + n_{22}$, we have $n_{12} + n_{21} = n_{21} + n_{12}$. If $n_{12} \leq n_{21}$, then

$$\begin{array}{c|cc} a_1 & n_{11} + n_{12} & 0 \\ a_2 & -n_{12} + n_{21} & n_{12} + n_{22} \\ \hline & b_1 & b_2 \end{array}$$

and this is also an RDP table for $a_1 + a_2 = b_1 + b_2$. But in such a case, $a_1 \leq b_1$ which is a contradiction. In a similar way we can prove that $n_{21} \not\leq n_{12}$. Hence, $n_{12} \parallel n_{21}$.

Similarly, if (m_{ij}) is a decomposition, then in the same way as for (n_{ij}) we have $m_{12} > 0$ and $m_{21} > 0$. \square

3 Riesz Decomposition Properties of the Lexicographic Product

In this section, we concentrate to the Riesz Decomposition Properties of the lexicographic product of two po-groups. In particular, we introduce the Com-Directness Property for po-groups. We start with the following result which was proved in [Dvu3, Thm 3.3].

Theorem 3.1. *Let A be an antilattice po-group and G be a directed po-group. Then $A \overrightarrow{\times} G$ satisfies RDP if and only if both A and G satisfy RDP.*

We note that we do not know whether Theorem 3.1 holds without assumption A is an antilattice po-group. In what follows we extend Theorem 3.1.

It is very interesting to mention that in Lemma 2.3, the po-group $A = \mathbb{R} \times \mathbb{R}$ is an Abelian antilattice po-group with RDP and as well as RDP₁, but as we have seen, if G is a directed non-Abelian po-group with RDP₁, then $A \overrightarrow{\times} G$ has RDP but RDP₁ fails. So the assumption that A is an antilattice is not a guarantee to be $A \overrightarrow{\times} G$ with RDP₁ if G has RDP₁.

We remind that if A is a linearly ordered group and G is a directed po-group with RDP₁, then $A \overrightarrow{\times} G$ satisfies RDP₁, see [Dvu3, Thm 3.3].

The following result is motivated by Lemma 2.3.

Theorem 3.2. *Let $\{A_i \mid i \in I\}$ be a system of non-trivial linearly ordered groups such that, for each $i \in I$, if $a \in A_i$, such that $a > 0$, then there is a_0 in A_i with $0 < a_0 < a$. Let the direct product*

$A = \prod_{i \in I} A_i$ be endowed with the strict product ordering, and G be a directed po-group with RDP. Then $A \overrightarrow{\times} G$ has RDP whenever every A_i is Abelian.

If G is a non-Abelian po-group with RDP₁ and every A_i is Abelian, then $A \overrightarrow{\times} G$ has RDP but RDP₁ fails if $|I| > 1$.

Proof. First we show that $A = \prod_{i \in I} A_i$ is an antilattice po-group with RDP. Let $c = (c^i)_{i \in I} \in A$ be the infimum of two mutually non-comparable elements $a = (a^i)_{i \in I} \in A$ and $b = (b^i)_{i \in I} \in A$. Then $c^i < a^i, b^i$ for each i . Since a^i and b^i are comparable, there is an element d^i such that $c^i < d^i < a^i, b^i$ which says $c < (d^i)_{i \in I} < a, b$, an absurd. Hence, A is an antilattice.

Assume every A_i is Abelian. Let $a = (a^i)_{i \in I} \in A^+, b = (b^i)_{i \in I} \in A^+, c = (c^i)_{i \in I} \in A^+$, and $d = (d^i)_{i \in I} \in A^+$ be such that

$$a + b = c + d. \quad (3.0)$$

Then every $a^i, b^i, c^i, d^i \geq 0$. If one of a, b, c, d is zero, an RDP table for (3.0) is evident. So assume that every a^i, b^i, c^i, d^i is strictly positive for each $i \in I$. Then using RDP in each A_i , we have an RDP table

$$\begin{array}{c|cc} a^i & n_{11}^i & n_{12}^i \\ b^i & n_{21}^i & n_{22}^i \\ \hline & c^i & d^i \end{array} (A).$$

If, say $n_{11}^i = 0$, then $n_{12}^i > 0$ and $n_{21}^i > 0$. Since n_{12}^i and n_{21}^i are ordered, there is $n_0^i \in A_i$ such that $0 < n_0^i < n_{12}^i, n_{21}^i$. Then in the table

$$\begin{array}{c|cc} a^i & n_{11}^i + n_0^i & n_{12}^i - n_0^i \\ b^i & n_{21}^i - n_0^i & n_{22}^i + n_0^i \\ \hline & c^i & d^i \end{array}$$

all entries are strictly positive, so that we can assume that all entries in (A) are strictly positive.

$$\begin{array}{c|cc} (a^i)_{i \in I} & (n_{11}^i)_{i \in I} & (n_{12}^i)_{i \in I} \\ (b^i)_{i \in I} & (n_{21}^i)_{i \in I} & (n_{22}^i)_{i \in I} \\ \hline & (c^i)_{i \in I} & (d^i)_{i \in I} \end{array}$$

is an RDP table for (2.1).

Since A is an antilattice with RDP, by [Dvu3, Thm. 3.3], $A \overrightarrow{\times} G$ has RDP.

Now assume that G is a non-Abelian po-group with RDP₁ and let $|I| > 1$. By the first part of the present proof, $A \overrightarrow{\times} G$ has RDP. In what follows, we show that RDP₁ fails.

Since, every A_i is non-trivial, it has infinitely many elements, and the index set has at least two elements, fix A_1, A_2 , and take two fixed elements $0 < a \in A_1$ and $0 < b \in A_2$. Then for the strictly positive elements $(1a, 4b), (3a, 7b), (2a, 3b), (2a, 8b)$, we have $(1a, 4b) + (3a, 7b) = (2a, 3b) + (2a, 8b)$. Similarly, as in the proof of Lemma 2.3, take two elements $x, y \in G$ such that $x + y \neq y + x$. Define positive elements $((a^i)_{i \in I}, x), ((b^i)_{i \in I}, y), ((c^i)_{i \in I}, x)$ and $((d^i)_{i \in I}, y)$ in $(A \overrightarrow{\times} G)^+$ such that $a^1 = 1a, a^2 = 4b, b^1 = 3a, b^2 = 7b, c^1 = 2a, c^2 = 3b$, and $d^1 = 2a, d^2 = 8b$.

Now we show that any RDP table

$$\begin{array}{c|cc} ((a^i)_{i \in I}, x) & ((n_{11}^i)_{i \in I}, c_{11}) & ((n_{12}^i)_{i \in I}, c_{12}) \\ ((b^i)_{i \in I}, y) & ((n_{21}^i)_{i \in I}, c_{21}) & ((n_{22}^i)_{i \in I}, c_{22}) \\ \hline & ((c^i)_{i \in I}, x) & ((d^i)_{i \in I}, y) \end{array} (B)$$

for $((a^i)_{i \in I}, x) + ((b^i)_{i \in I}, y) = ((c^i)_{i \in I}, x) + ((d^i)_{i \in I}, y)$ gives no RDP₁ table. In fact, we have two kinds of equations

$$((1a, 4b), x) + ((3a, 7b), y) = ((2a, 3b), x) + ((2a, 8b), y)$$

and

$$a^i + b^i = c^i + d^i$$

for each $i \in I \setminus \{1, 2\}$.

For the first one we have from (B) the following RDP table

$((1a, 4b), x)$	$((n_{11}^1, n_{11}^2, c_{11})$	$((n_{12}^1, n_{12}^2), c_{12})$
$((3a, 7b), y)$	$((n_{21}^1, n_{21}^2), c_{21})$	$((n_{22}^1, n_{22}^2), c_{22})$
	$((2a, 3b), x)$	$((2a, 8b), y)$

The elements $(3a, 7b)$ and $(2a, 3b)$ are non-comparable as well as are $(1a, 4b)$ and $(2a, 8b)$. Therefore, $n_{12}^1, n_{12}^2, n_{21}^1, n_{21}^2 > 0$. There are non-zero elements $s \in A_1$ and $t \in A_2$ such that $0 < s < n_{12}^1, n_{21}^1$ and $0 < t < n_{12}^2, n_{21}^2$. Hence $0 \leq ((s, t), x) \leq (n_{12}^1, n_{12}^2), c_{12})$ and $((s, t), x) \leq (n_{21}^1, n_{21}^2), c_{21})$ but $((s, t), x) + ((s, t), y) \neq ((s, t), y) + ((s, t), x)$.

Therefore, RDP_1 fails for $A \vec{\times} G$. \square

Theorem 3.3. Let $\{A_i \mid i \in I\}$ be a family of linearly ordered po-groups and G be a directed Abelian po-group with RDP_1 . Then $(\prod_{i \in I} A_i) \vec{\times} G$ has RDP_1 .

Proof. Let elements $((a^i)_{i \in I}, x), ((b^i)_{i \in I}, y), ((c^i)_{i \in I}, z), ((d^i)_{i \in I}, u) \in (\prod_{i \in I} A_i \vec{\times} G)^+$ satisfy $((a^i)_{i \in I}, x) + ((b^i)_{i \in I}, y) = ((c^i)_{i \in I}, z) + ((d^i)_{i \in I}, u)$. Then clearly, $(a^i)_{i \in I}, (b^i)_{i \in I}, (c^i)_{i \in I}, (d^i)_{i \in I} \geq (0)_{i \in I}$. Since every A_i is linearly ordered, A_i is an ℓ -group, so it satisfies RDP_2 and RDP_1 .

(1) If $(a^i)_{i \in I} < (c^i)_{i \in I}$, then for each $i \in I$, $a^i \leq c^i$ and there exists $j \in I$ such that $a_j < c_j$ and so we have the following RDP_1 table:

$((c^i)_{i \in I}, z)$	$((a^i)_{i \in I}, x)$	$((-a^i + c^i)_{i \in I}, -x + z)$
$((d^i)_{i \in I}, u)$	$((0)_{i \in I}, 0)$	$((d^i)_{i \in I}, u)$
	$((a^i)_{i \in I}, x)$	$((b^i)_{i \in I}, y)$

(2) If $(a^i)_{i \in I} > (c^i)_{i \in I}$, then similarly to (1) we can find an RDP_1 table.

(3) If $(a^i)_{i \in I} = (c^i)_{i \in I}$, then clearly $(b^i)_{i \in I} = (d^i)_{i \in I}$.

(i) If $(a^i)_{i \in I} = (0)_{i \in I}$ and $(b_i)_{i \in I} = (0)_{i \in I}$, then clearly we can find an RDP_1 table.

(ii) If $(a^i)_{i \in I} = (0)_{i \in I}$ and $(b_i)_{i \in I} \neq (0)_{i \in I}$, then we have $x, z \geq 0$. Let t be a lower bound for y, u . Consider an RDP_1 decomposition for $x + (y - t) = z + (u - t)$ as follows:

z	c_{11}	c_{12}
$u - t$	c_{21}	c_{22}
	x	$y - t$

Then we have

z	c_{11}	c_{12}
u	c_{21}	$c_{22} + t$
	x	y

(*)

and so

$((c^i)_{i \in I}, z)$	$((0)_{i \in I}, c_{11})$	$((0)_{i \in I}, c_{12})$
$((d^i)_{i \in I}, u)$	$((0)_{i \in I}, c_{21})$	$((b^i)_{i \in I}, c_{22} + t)$
	$((a^i)_{i \in I}, x)$	$((b^i)_{i \in I}, y)$

is an RDP_1 table for $((a^i)_{i \in I}, x) + ((b^i)_{i \in I}, y) = ((c^i)_{i \in I}, z) + ((d^i)_{i \in I}, u)$.

(iii) If $(a^i)_{i \in I} \neq (0)_{i \in I}$ and $(b_i)_{i \in I} = (0)_{i \in I}$, then similarly to (iii) we can find an RDP_1 table for $((a^i)_{i \in I}, x) + ((b^i)_{i \in I}, y) = ((c^i)_{i \in I}, z) + ((d^i)_{i \in I}, u)$.

(iv) Let $(a^i)_{i \in I} \neq (0)_{i \in I}$ and $(b_i)_{i \in I} \neq (0)_{i \in I}$. Since G is directed, there exists $t \in G$ such that $t \leq x, y, z, u$. Let

$(-t + z)$	c_{11}	c_{12}
$(u - t)$	c_{21}	c_{22}
	$(-t + x)$	$(y - t)$

be an RDP_1 table for $(-t+z) + (u-t) = (-t+x) + (y-t)$. Then we have

$$\begin{array}{c|cc} z & t+c_{11} & c_{12} \\ u & c_{21} & c_{22}+t \\ \hline & x & y \end{array}. \quad (**)$$

It gives an RDP_1 table for $((a^i)_{i \in I}, x) + ((b^i)_{i \in I}, y) = ((c^i)_{i \in I}, z) + ((d^i)_{i \in I}, u)$

$$\begin{array}{c|cc} ((c^i)_{i \in I}, z) & ((a^i)_{i \in I}, t+c_{11}) & ((0)_{i \in I}, c_{12}) \\ ((d^i)_{i \in I}, u) & ((0)_{i \in I}, c_{21}) & ((d^i)_{i \in I}, c_{22}+t) \\ \hline & ((a^i)_{i \in I}, x) & ((b^i)_{i \in I}, y) \end{array}.$$

(4) Let $(c^i)_{i \in I}$ and $(a^i)_{i \in I}$ be not comparable. Consider the following subsets of I

$$I_1 := \{i \in I \mid a^i < c^i\}, \quad I_2 := \{i \in I \mid c^i < a^i\}, \quad I_3 := \{i \in I \mid a^i = c^i\}.$$

By the assumption, $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$, and if $i \in I_3$, then $d^i = b^i$. Set

$$e^i = \begin{cases} a^i & \text{if } i \in I_1 \\ c^i & \text{if } i \in I_2 \\ a^i & \text{if } i \in I_3 \end{cases} \quad f^i = \begin{cases} -a^i + c^i & \text{if } i \in I_1 \\ 0 & \text{if } i \in I_2 \\ 0 & \text{if } i \in I_3 \end{cases} \quad (3.1)$$

$$g^i = \begin{cases} 0 & \text{if } i \in I_1 \\ -c^i + a^i & \text{if } i \in I_2 \\ 0 & \text{if } i \in I_3 \end{cases} \quad h^i = \begin{cases} d^i & \text{if } i \in I_1 \\ b^i & \text{if } i \in I_2 \\ d^i & \text{if } i \in I_3. \end{cases} \quad (3.2)$$

Since G is directed, there exists a $d \in G$ such that $d \leq x, y, z, u$. Let

$$\begin{array}{c|cc} -d+z & c_{11} & c_{12} \\ u-d & c_{21} & c_{22} \\ \hline & -d+x & y-d \end{array}$$

be an RDP_1 table for $(-d+x) + (y-d) = (-d+z) + (u-d)$. Since G is Abelian, we have

$$\begin{array}{c|cc} z & c_{11} & c_{12}+d \\ u & c_{21}+d & c_{22} \\ \hline & x & y \end{array}.$$

Then

$$\begin{array}{c|cc} ((c^i)_{i \in I}, z) & ((e^i)_{i \in I}, c_{11}) & ((f^i)_{i \in I}, c_{12}+d) \\ ((d^i)_{i \in I}, u) & ((g^i)_{i \in I}, c_{21}+d) & ((h^i)_{i \in I}, c_{22}) \\ \hline & ((a^i)_{i \in I}, x) & ((b^i)_{i \in I}, y) \end{array}$$

is an RDP_1 table for $((a^i)_{i \in I}, x) + ((b^i)_{i \in I}, y) = ((c^i)_{i \in I}, z) + ((d^i)_{i \in I}, u)$. Indeed, let $((0)_{i \in I}, 0) \leq ((k^i)_{i \in I}, w) \leq ((f^i)_{i \in I}, c_{12}+d)$ and $((0)_{i \in I}, 0) \leq ((m^i)_{i \in I}, v) \leq ((g^i)_{i \in I}, c_{21}+d)$. Then $(0)_{i \in I} \leq (k^i)_{i \in I} \leq (f^i)_{i \in I}$ and $(0)_{i \in I} \leq (m^i)_{i \in I} \leq (g^i)_{i \in I}$. If $i \in I - I_1$, then $f^i = 0$ and so $k^i = 0$. That is, $k^i = 0$ for all $i \in I - I_1$. Similarly, $m^i = 0$ for all $i \in I - I_2$. Put $i \in I$. If $m^i \neq 0$, then $i \in I_2$, so $k^i = 0$. It follows that $m^i + k^i = k^i + m^i$. For $m^i = 0$ clearly, $m^i + k^i = k^i + m^i$. Thus $(m^i)_{i \in I} + (k^i)_{i \in I} = (k^i)_{i \in I} + (m^i)_{i \in I}$. Since G is Abelian, then we have $((m^i)_{i \in I}, v) + ((k^i)_{i \in I}, w) = ((k^i)_{i \in I}, w) + ((m^i)_{i \in I}, v)$.

From (1)–(4) we get that $(\prod_{i \in I} A_i) \xrightarrow{\sim} G$ has RDP_1 . \square

4 Com-Directness Property and Lexicographic Product

In this section we show that if we put more general conditions to the second factor, G , of the lexicographic product $A \overrightarrow{\times} G$, than G is Abelian, we can extend the class of po-groups with RDP such that $A \overrightarrow{\times} G$ has RDP for each po-group A with RDP. Such a condition is the com-directness of G .

For the aims of the following theorem, we introduce a stronger form of the directness of a po-group which is motivated as follows. If G is a po-group, then for each $x, y \in G^+$, there is $d \in G$ such that $d \leq x, y$ and d commutes with x and with y ; in this case such d can be trivially used $d = 0$. The same is true for a directed Abelian po-group. Therefore, we say that a po-group G is *com-directed* (com stands for the commutativity), if given $x, y \in G$, there is a $d \in Z(G)$, where $Z(G)$ is the center of G , such that $d \leq x, y$. This is equivalent, given $x, y \in G$, there is a $d \in Z(G)$ such that $x, y \leq d$. Of course, if G is com-directed, then G is directed, too. If G is Abelian, then both notions, directness and com-directness, coincide.

We note that if a po-group G is com-directed, then it does not follow that G is Abelian. Indeed, let G be a po-group that is not Abelian. Then $\mathbb{Z} \overrightarrow{\times} G$ is a com-directed po-group that is not Abelian; indeed, given $(n, g), (m, h) \in \mathbb{Z} \overrightarrow{\times} G$, any element $d = (k, 0) \in \mathbb{Z} \overrightarrow{\times} G$, where $k < n, m$, is from the center $Z(\mathbb{Z} \overrightarrow{\times} G)$, and we have $d < (n, g), (m, h)$.

Theorem 4.1. *Let $(A; +, 0)$ be a po-group and $(G; +, 0)$ be a com-directed po-group. Then $A \overrightarrow{\times} G$ satisfies RDP if and only if both A and G satisfy RDP.*

Proof. Let $A \overrightarrow{\times} G$ satisfy RDP, and let $x_1 + x_2 = y_1 + y_2$ in A for $x_1, x_2, y_1, y_2 \in A^+$. Then $(x_1, 0) + (x_2, 0) = (y_1, 0) + (y_2, 0)$ which easily implies that $x_1 + x_2 = y_1 + y_2$ has an RDP table in A . If now for $u_1, u_2, v_1, v_2 \in G^+$ we have $u_1 + u_2 = v_1 + v_2$, then $(0, u_1) + (0, u_2) = (0, v_1) + (0, v_2)$. The RDP in $A \overrightarrow{\times} G$ implies that $u_1 + u_2 = v_1 + v_2$ has an RDP table in G .

Now let A and G have RDP. If G is the trivial po-group, i.e. $G = \{0\}$, then $A \overrightarrow{\times} G = A \overrightarrow{\times} \{0\} \cong A$ and $A \overrightarrow{\times} G$ satisfies RDP.

Let us assume that G is a non-trivial com-directed po-group with RDP. Let us have elements $(x_1, u_1), (x_2, u_2), (y_1, v_1), (y_2, v_2) \in (A \overrightarrow{\times} G)^+$ such that

$$(x_1, u_1) + (x_2, u_2) = (y_1, v_1) + (y_2, v_2). \quad (4.1)$$

(I) First we assume that x_1 and y_1 are comparable. In view of (4.1), x_2 and y_2 are also comparable. We have the following 9 subcases.

(i) Let $(0, u_1) + (0, u_2) = (0, v_1) + (0, v_2)$. Then $u_1, u_2, v_1, v_2 \in G^+$ and RDP for this case follows from RDP for G .

(ii) $(0, u_1) + (x, u_2) = (0, v_1) + (y, v_2)$ for $u_1, v_1 \geq 0$, $u_2, v_2 \in G$ for each $x = y \in A^+ \setminus \{0\}$. Then $u_1 + u_2 = v_1 + v_2$. While G is directed, there is an element $d \in G$ such that $u_2, v_2 \geq d$. Then $u_1 + (u_2 - d) = v_1 + (v_2 - d)$ and for them we have an RDP decomposition

$$\begin{array}{c|cc} u_1 & c_{11} & c_{12} \\ u_2 - d & c_{21} & c_{22} \\ \hline & v_1 & v_2 - d \end{array}.$$

Then

$$\begin{array}{c|cc} u_1 & c_{11} & c_{12} \\ u_2 & c_{21} & c_{22} + d \\ \hline & v_1 & v_2 \end{array}$$

and

$$\begin{array}{c|cc} (0, u_1) & (0, c_{11}) & (0, c_{12}) \\ (x, u_2) & (0, c_{21}) & (y, c_{22} + d) \\ \hline & (0, v_1) & (y, v_2) \end{array}$$

is an RDP decomposition for (ii) in the po-group $A \overrightarrow{\times} G$.

(iii) $(x, u_1) + (0, u_2) = (y, v_1) + (0, v_2)$ for $u_2, v_2 \geq 0$, $u_1, v_1 \in G$ for $x = y \in A^+ \setminus \{0\}$. The directness of G implies, there is $d \in G$ such that $d \leq u_1, u_2, v_1, v_2$. Equality (iii) can be rewritten in the equivalent form $(x, -d + u_1) + (0, u_2 - d) = (y, -d + v_1) + (0, v_2 - d)$ which yields $(-d + u_1) + (u_2 - d) = (-d + v_1) + (v_2 - d)$. It entails an RDP decomposition in the po-group G

$$\begin{array}{c|cc} -d + u_1 & c_{11} & c_{12} \\ u_2 - d & c_{21} & c_{22} \\ \hline & -d + v_1 & v_2 - d \end{array},$$

consequently,

$$\begin{array}{c|cc} u_1 & d + c_{11} & c_{12} \\ u_2 & c_{21} & c_{22} + d \\ \hline & v_1 & v_2 \end{array},$$

and it gives an RDP decomposition of (iii) in the po-group $A \overrightarrow{\times} G$

$$\begin{array}{c|cc} (x, u_1) & (x, d + c_{11}) & (0, c_{12}) \\ (0, u_2) & (0, c_{21}) & (0, c_{22} + d) \\ \hline & (y, v_1) & (0, v_2) \end{array}.$$

(iv) $(x, u_1) + (0, u_2) = (0, v_1) + (y, v_2)$ for $u_1, v_2 \in G$, $u_2, v_1 \geq 0$ for $x = y \in A^+ \setminus \{0\}$.

Then $u_1 + u_2 = v_1 + v_2$ which implies $-v_1 + u_1 = v_2 - u_2$. If we use the decomposition

$$\begin{array}{c|cc} (x, u_1) & (0, v_1) & (x, -v_1 + u_1) \\ (0, u_2) & (0, 0) & (0, u_2) \\ \hline & (0, v_1) & (y, v_2) \end{array},$$

we see that it gives an RDP decomposition for (iv).

(v) $(x, u_1) + (0, u_2) = (y_1, v_1) + (y_2, v_2)$ for $u_1, v_1, v_2 \in G$, $u_2 \geq 0$, where $y_1, y_2 \in A^+ \setminus \{0\}$ and $y_1 + y_2 = x$. Then $u_1 + u_2 = v_1 + v_2$. Hence, the following table gives an RDP decomposition for (v)

$$\begin{array}{c|cc} (x, u_1) & (y_1, v_1) & (y_2, -v_1 + u_1) \\ (0, u_2) & (0, 0) & (0, u_2) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array}.$$

(vi) $(0, u_1) + (x, u_2) = (y_1, v_1) + (y_2, v_2)$ for $u_2, v_1, v_2 \in G$, $u_1 \geq 0$, where $y_1, y_2 \in A^+ \setminus \{0\}$ and $y_1 + y_2 = x$. Then we have $v_1 + v_2 = u_1 + u_2$ and the following RDP decomposition

$$\begin{array}{c|cc} (0, u_1) & (0, u_1) & (0, 0) \\ (x, u_2) & (y_1, -u_1 + v_1) & (y_2, v_2) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array}.$$

(vii) $(x_1, u_1) + (x_2, u_2) = (y_1, v_1) + (y_2, v_2)$ for $u_1, u_2, v_1, v_2 \in G$, where $x_1, x_2, y_1, y_2 \in A^+ \setminus \{0\}$, $x_1 + x_2 = y = y_1 + y_2$ and $x_1 > y_1$. Then $u_1 + u_2 = v_1 + v_2$, and since $-y_1 + y = y_2$, (vii) has the following RDP decomposition

$$\begin{array}{c|cc} (x_1, u_1) & (y_1, v_1) & (-y_1 + x_1, -v_1 + u_1) \\ (x_2, u_2) & (0, 0) & (x_2, u_2) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array} \text{ if } x_1 > y_1$$

is an RDP decomposition.

(viii) $(x_1, u_1) + (x_2, u_2) = (y_1, v_1) + (y_2, v_2)$ for $u_1, u_2, v_1, v_2 \in G$, where $x_1, x_2, y_1, y_2 \in A^+ \setminus \{0\}$, $x_1 + x_2 = y = y_1 + y_2$ and $y_1 > x_1$. Then (viii) follows from (vii) when we rewrite (viii) in the equivalent form $(y_1, v_1) + (y_2, v_2) = (x_1, u_1) + (x_2, u_2)$, and an RDP table is as follows

$$\begin{array}{c|cc} (x_1, u_1) & (y_1, v_1) & (-y_1 + x_1, -v_1 + u_1) \\ (x_2, u_2) & (0, 0) & (x_2, u_2) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array} \text{ if } y_1 > x_1.$$

(ix) $(x_1, u_1) + (x_2, u_2) = (y_1, v_1) + (y_2, v_2)$ for $u_1, u_2, v_1, v_2 \in G_2$, where $x_1, x_2, y_1, y_2 \in A^+ \setminus \{0\}$, $x_1 + x_2 = x = y_1 + y_2$ and $x_1 = y_1$. Then $u_1 + u_2 = v_1 + v_2$. The directness of G entails that there is $d \in G$ such that $u_1, u_2, v_1, v_2 \geq d$. Hence, $(-d + u_1) + (u_2 - d) = (-d + v_1) + (v_2 - d)$. The RDP holding in G entails the following RDP table

$$\begin{array}{c|cc} -d + u_1 & c_{11} & c_{12} \\ u_2 - d & c_{21} & c_{22} \\ \hline & -d + v_1 & v_2 - d \end{array},$$

so that

$$\begin{array}{c|cc} u_1 & d + c_{11} & c_{12} \\ u_2 & c_{21} & c_{22} + d \\ \hline & v_1 & v_2 \end{array}.$$

It gives an RDP decomposition of (ix)

$$\begin{array}{c|cc} (x_1, u_1) & (x_1, d + c_{11}) & (0, c_{12}) \\ (x_2, u_2) & (0, c_{21}) & (x_2, c_{22} + d) \\ \hline & (x_1, v_1) & (x_2, v_2) \end{array}.$$

(II) Let x_1 and y_1 be not comparable. In particular, we have $x_1 > 0$ and $y_1 > 0$. Since in A we have the RDP property, from $x_1 + x_2 = y_1 + y_2$ we have an RDP table

$$\begin{array}{c|cc} x_1 & e_{11} & e_{12} \\ x_2 & e_{21} & e_{22} \\ \hline & y_1 & y_2 \end{array}.$$

By the assumptions, there is $d \in Z(G)$ such that $d \leq u_1, u_2, v_1, v_2$. Hence, for the equality of positive elements $(-d + u_1) + (u_2 - d) = (-d + v_1) + (v_2 - d)$ in G , there is an RDP table

$$\begin{array}{c|cc} -d + u_1 & c_{11} & c_{12} \\ u_2 - d & c_{21} & c_{22} \\ \hline & -d + v_1 & v_2 - d \end{array},$$

which entails

$$\begin{array}{c|cc} u_1 & d + c_{11} & c_{12} \\ u_2 & c_{21} & c_{22} + d \\ \hline & v_1 & v_2 \end{array}.$$

If $e_{11} > 0$ and $e_{22} > 0$, then

$$\begin{array}{c|cc} (x_1, u_1) & (e_{11}, d + c_{11}) & (e_{12}, c_{12}) \\ (x_2, u_2) & (e_{21}, c_{21}) & (e_{22}, c_{22} + d) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array}$$

is an RDP table for (4.1).

If $e_{11} = 0$, then $e_{12} > 0$ and $e_{21} > 0$ (otherwise, x_1 and y_1 are comparable). Then the following table gives an RDP decomposition for (4.1)

$$\begin{array}{c|cc} (x_1, u_1) & (e_{11}, c_{11}) & (e_{12}, d + c_{12}) \\ (x_2, u_2) & (e_{21}, c_{21} + d) & (e_{22}, c_{22}) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array}.$$

If $e_{22} = 0$, then $e_{12} > 0$ and $e_{21} > 0$, and the last RDP table is an RDP table also for this case. Summing up all above cases, we see that $A \overrightarrow{\times} G$ satisfies RDP. \square

Remark 4.2. (i) We note that the com-directness property of G was used only in the last table, and in all the other cases, we have used only the assumption G is directed. Therefore, there is a natural question, does Theorem 4.1 hold assuming G is not necessarily com-directed and rather directed?

(ii) If G is a directed Abelian po-group with RDP, then G is com-directed and $A \overrightarrow{\times} G$ has RDP for each po-group A with RDP.

(iii) Theorem 4.1 does not hold in the RDP_1 variant, in general. Indeed, let $G = \mathbb{Z} \overrightarrow{\times} H$, where H is a directed non-Abelian po-group with RDP_1 . By the note just before Theorem 4.1, it was shown that G is a com-directed non-Abelian po-group, and applying [DvKo, Thm 3.3], we get G has RDP_1 . Using Lemma 2.3, we see that if $A = \mathbb{R} \times \mathbb{R}$ is with strict product ordering, then the Abelian po-group A has both RDP and RDP_1 , but $A \overrightarrow{\times} G$ has only RDP and RDP_1 fails in it.

Remark 4.3. A partial answer for the latter note (i) is the assumption that A satisfies the following condition:

Given $a, b \in A^+ \setminus \{0\}$ such that a and b are not comparable, there is $d' \in A$, $0 < d' \leq a, b$ such that $a + d' = d' + a$ and $b + d' = d' + b$.

Hence, if G is directed, and A and G satisfy RDP, then $A \overrightarrow{\times} G$ has RDP.

Proof. It is enough to verify the last case of (II) in the proof of Theorem 4.1.

Since $e_{12} > 0$ and $e_{21} > 0$ are not comparable, there is $0 < d' \in A$ such that $d' \leq e_{12}, e_{21}$ and d' commutes with e_{21} and e_{12} . Then the following table

$$\begin{array}{c|cc} (x_1, u_1) & (e_{11} + d', d + c_{11}) & (-d' + e_{12}, c_{12}) \\ (x_2, u_2) & (-d' + e_{21}, c_{21}) & (d' + e_{22}, c_{22} + d) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array}$$

is an RDP table for the last case of (II) in the proof of Theorem 4.1. \square

Remark 4.3 can be generalized as follows.

Remark 4.4. Let A be a po-group with RDP such that, given two non-comparable elements $a, b \in A^+ \setminus \{0\}$, there is an element $d' \in A$ such that $0 < d' \leq a, b$ and $-a + d' + a = -b + d' + b$. If G satisfies RDP, then $A \overrightarrow{\times} G$ has RDP.

Proof. It is enough to verify the last case of (II) in the proof of Theorem 4.1.

Since $e_{12} > 0$ and $e_{21} > 0$ are not comparable, there is $0 < d' \in A$ such that $d' \leq e_{12}, e_{21}$ and $-e_{12} + d' + e_{12} = -e_{21} + d' + e_{21}$. Then the following table

$$\begin{array}{c|cc} (x_1, u_1) & (e_{11} + d', d + c_{11}) & (-d' + e_{12}, c_{12}) \\ (x_2, u_2) & (-d' + e_{21}, c_{21}) & (-e_{12} + d' + e_{12} + e_{22}, c_{22} + d) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array}$$

is an RDP table for the last case of (II) in the proof of Theorem 4.1. Indeed, we have $-d' + e_{21} + (-e_{12} + d' + e_{12} + e_{22}) = -d' + e_{21} + (-e_{21} + d' + e_{21} + e_{22}) = e_{21} + e_{22} = x_2$. \square

5 Non-Comparability Directness Property and Lexicographic Product

We continue with study of the Riesz Decomposition Properties of the lexicographic product of po-groups. In particular, we introduce the Non-Comparability Directness Property, and some illustrating examples will be present.

A po-group A satisfying the condition “given two non-comparable elements $a, b \in A^+ \setminus \{0\}$, there is an element $d \in A$ such that $0 < d \leq a, b$ and $-a + d + a = -b + d + b$ ” is said to be a po-group with (or satisfying) the *Non-Comparability Directness Property*, NCDP for short.

Theorem 5.1. (i) Let A be a po-group with RDP such that, given two non-comparable elements $a, b \in A^+ \setminus \{0\}$ with $a + b = b + a$, there is an element $d' \in A$ such that $0 < d' \leq a, b$ and $-a + d' + a = -b + d' + b$. If a po-group G satisfies RDP, then $A \overrightarrow{\times} G$ has RDP.

(ii) Let A be a po-group with RDP and G be a directed po-group. Let A do not satisfy the condition in part (i) and let $A \overrightarrow{\times} G$ have RDP, then G has RDP and, for each equation $u_1 + u_2 = v_1 + v_2$ of elements of G , there exist $d_1, d_2 \in G$ such that

$$(1) \quad d_1 \leq u_1, v_2 \text{ and } d_2 \leq u_2, v_1;$$

$$(2) \quad d_1 + d_2 = d_2 + d_1;$$

$$(3) \quad -u_1 + v_1 = -d_1 + d_2.$$

(iii) Let G have RDP. If, for each equation $u_1 + u_2 = v_1 + v_2$ of elements of a po-group G , there exist $d_1, d_2 \in G$ such that conditions (1)–(3) of (ii) are satisfied, then G is directed and $A \overrightarrow{\times} G$ has RDP for each po-group A with RDP.

Proof. (i) It is enough to verify the last case of (II) in the proof of Theorem 4.1. From $x_1 + x_2 = y_1 + y_2$ it follows that $e_{11} + e_{12} + e_{21} + e_{22} = e_{11} + e_{21} + e_{12} + e_{22}$ and so $e_{12} + e_{21} = e_{21} + e_{12}$. Since $e_{12} > 0$ and $e_{21} > 0$ are not comparable, there is $0 < d' \in A$ such that $d' \leq e_{12}, e_{21}$ and $-e_{12} + d' + e_{12} = -e_{21} + d' + e_{21}$. Then the following RDP table

$$\begin{array}{c|cc} (x_1, u_1) & (e_{11} + d', d + c_{11}) & (-d' + e_{12}, c_{12}) \\ (x_2, u_2) & (-d' + e_{21}, c_{21}) & (-e_{12} + d' + e_{12} + e_{22}, c_{22} + d) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array}$$

is for the last case of (II) in the proof of Theorem 4.1. Indeed, we have $-d' + e_{21} + (-e_{12} + d' + e_{12} + e_{22}) = -d' + e_{21} + (-e_{21} + d' + e_{21} + e_{22}) = e_{21} + e_{22} = x_2$.

(ii) By the assumption there are non-comparable $a, b \in A^+ \setminus \{0\}$ with $a + b = b + a$ such that there is no $0 < d \leq a, b$ satisfying the condition $-a + d - a = -b + d + b$. First we assume that $A \overrightarrow{\times} G$ has RDP. Then clearly G has RDP. Choose arbitrary elements $u_1, u_2, v_1, v_2 \in G$ such that $u_1 + u_2 = v_1 + v_2$. Then $(a, u_1), (b, u_2), (b, v_1), (a, v_2)$ are positive elements of $A \overrightarrow{\times} G$ and $(a, u_1) + (b, u_2) = (b, v_1) + (a, v_2)$ and so we have an RDP table in $A \overrightarrow{\times} G$ as follows

$$\begin{array}{c|cc} (a, u_1) & (e_{11}, c_{11}) & (e_{12}, c_{12}) \\ (b, u_2) & (e_{21}, c_{21}) & (e_{22}, c_{22}) \\ \hline & (b, v_1) & (a, v_2) \end{array}.$$

Clearly, $0 \leq e_{ij}$ for all $i, j \in \{1, 2\}$. If $e_{11} > 0$, then $e_{12} < a$ (since $a = e_{11} + e_{12}$). Also, $-a + e_{11} + a = -e_{12} - e_{11} + e_{11} + e_{12} + e_{22} = e_{22}$ and $-b + e_{11} + b = -e_{21} - e_{11} + e_{11} + e_{21} + e_{22} = e_{22}$, so $-a + e_{11} + a = -b + e_{11} + b$ which is a contradiction. Therefore, $e_{11} = 0$. In a similar way, $e_{22} = 0$. Thus c_{11} and c_{22} are positive elements of G . From $u_1 = c_{11} + c_{12}$ and $v_2 = c_{12} + c_{22}$, we get that $c_{12} \leq u_1, v_2$. Similarly, $c_{21} \leq u_2, v_1$. Clearly, $c_{12} + c_{21} = c_{21} + c_{12}$. Moreover, $-u_1 + v_1 = -c_{12} - c_{11} + c_{11} + c_{21} = -c_{12} + c_{21}$. Set $d_1 = e_{12}$ and $d_2 = e_{21}$.

(iii) Suppose that G has RDP and, for each equation $u_1 + u_2 = v_1 + v_2$ of elements of G , there are $d_1, d_2 \in G$ satisfying the above mentioned properties (1)–(3). First we show that G is directed. Indeed, let $u, v \in G$ be given. For the equality $u + (u + v) = 2u + v$, there is $d_1 \in G$ such that $d_1 \leq u, v$.

Now let A be a po-group with RDP. We show that $A \overrightarrow{\times} G$ has RDP. It is enough to verify the last case of (II) in the proof of Theorem 4.1. By the assumption, there are $d_1, d_2 \in G$ such that $d_1 \leq u_1, v_2$, $d_2 \leq u_2, v_1$, $d_1 + d_2 = d_2 + d_1$ and $-u_1 + v_1 = -d_1 + d_2$.

If $e_{11} = 0$ or $e_{22} = 0$, then $e_{12} > 0$ and $e_{21} > 0$, we claim that we have an RDP table

$$\begin{array}{c|cc} (x_1, u_1) & (e_{11}, u_1 - d_1) & (e_{12}, d_1) \\ (x_2, u_2) & (e_{21}, d_2) & (e_{22}, -d_2 + u_2) \\ \hline & (y_1, v_1) & (y_2, v_2) \end{array}$$

for $(x_1, u_1) + (x_2, u_2) = (y_1, v_1) + (y_2, v_2)$. Clearly, $(e_{11}, u_1 - d_1), (e_{12}, d_1), (e_{21}, d_2), (e_{22}, -d_2 + u_2)$ are positive elements of $A \overrightarrow{\times} G$. Also, $d_1 - d_2 + u_2 = -(-u_1 + v_1) + u_2 = -v_1 + u_1 + u_2 = v_2$ and $u_1 - d_1 + d_2 = u_1 - u_1 + v_1 = v_1$, so $A \overrightarrow{\times} G$ has RDP. \square

Inspired by (iii) of the latter theorem, we say that a po-group G has *wRDP*, (w stands for strong) if, for each equation $u_1 + u_2 = v_1 + v_2$ of elements of G , there exist $d_1, d_2 \in G$ such that conditions (1)–(3) of Theorem 5.1(ii) are satisfied.

Now we present an example of a directed non-Abelian po-group A with NCDP and RDP satisfying the assumptions of Theorem 5.1(i).

Example 5.2. Consider the po-group $B := (\mathbb{R} \times \mathbb{R}; +, (0, 0))$ with the strict product ordering. Clearly, B is directed and has RDP. Let $(C; +, 0)$ be a non-Abelian linearly ordered group. By [Dvu3, Thm. 3.1], $A = C \overrightarrow{\times} B$ has RDP. We claim that A satisfies the assumptions of Theorem 5.1(i). Put $(a_1, a_2), (b_1, b_2) \in A^+ \setminus \{0\}$ such that (a_1, a_2) and (b_1, b_2) are non-comparable. Since C is a chain, then $a_1 = b_1$ (otherwise, they are comparable), so a_2 and b_2 must be non-comparable. Let $a_2 = (x, y)$ and $b_2 = (u, v)$.

(i) Let $a_1 > 0$. Clearly, we can find an element $d \in \mathbb{R} \times \mathbb{R}$ such that d is strictly less than a_2 and b_2 and so (a_1, d) is a strictly positive element of $C \overrightarrow{\times} B$ and $(a_1, d) < (a_1, a_2), (b_1, b_2)$.

(ii) Let $a_1 = 0$. Then a_2 and b_2 are positive elements of B . Since B has the strict product ordering, it follows that $x, y, u, v > 0$. We can find $s, t \in \mathbb{R}$ such that $0 < s < x, u$ and $0 < t < y, v$. Clearly, $d = (s, t)$ is a strictly positive element of $\mathbb{R} \times \mathbb{R}$ and $d < a_2, b_2$. So, (a_1, d) is a strictly positive element of A which is strictly less than (a_1, a_2) and (b_1, b_2) .

In both cases we have $-(a_1, a_2) + (a_1, d) + (a_1, a_2) = (a_1, d) = -(b_1, b_2) + (a_1, d) + (b_1, b_2)$ which proves the claim.

The latter example can be strengthened as follows:

Example 5.3. If in Example 5.2, we assume that C is a linearly ordered non-trivial group such that $Z(A) = \{0\}$, then $A = C \overrightarrow{\times} B$ satisfies the conditions of Theorem 5.1(i) and is not com-directed.

Indeed, let $(a_1, a_2), (b_1, b_2)$ be two elements of A such that $a_1, b_1 < 0$. Since $Z(A) = \{0\} \times B$, there is no element (c_1, c_2) from $Z(A)$ such that $(c_1, c_2) \leq (a_1, a_2), (b_1, b_2)$.

An example of C satisfying our conditions is the class of square matrices of the form

$$A(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

for $a > 0, b \in (-\infty, \infty)$ with usual multiplication of matrices. It is a non-commutative linearly ordered group with the neutral element $A(1, 0)$ and with the positive cone consisting of matrices $A(a, b)$ with $a > 1$ or $a = 1$ and $b \geq 0$. For it we have $Z(C) = \{A(1, 0)\}$.

Example 5.4. Let H be a directed po-group with RDP. Then $H \overrightarrow{\times} \mathbb{R}$ satisfies the conditions in Theorem 5.1(i).

Proof. Since H is directed, $A = H \overrightarrow{\times} \mathbb{R}$ is directed. By Theorem 4.1, $H \overrightarrow{\times} \mathbb{R}$ has RDP. Let (a_1, a_2) and (b_1, b_2) be strictly positive elements of $H \overrightarrow{\times} \mathbb{R}$ such that $(a_1, a_2) + (b_1, b_2) = (b_1, b_2) + (a_1, a_2)$.

(1) If $0 < a_1, b_1$, then set $d = (0, 1)$. Clearly, $d > (0, 0)$ and $-(a_1, a_2) + d + (a_1, a_2) = (0, 1) = -(b_1, b_2) + d + (b_1, b_2)$.

(2) If $a_1 = b_1 = 0$, then $a_2, b_2 > 0$. Let $t \in \mathbb{R}$ such that $t < a_2, b_2$. Set $d = (0, t)$. We have $-(a_1, a_2) + d + (a_1, a_2) = (0, t) = -(b_1, b_2) + d + (b_1, b_2)$.

(3) If $a_1 = 0$ and $b_1 > 0$, then $a_2 > 0$. Set $d = (0, t)$, where $0 < t < a_2$. We have $0 < d < (a_1, a_2), (b_1, b_2)$ and $-(a_1, a_2) + d + (a_1, a_2) = (0, t) = -(b_1, b_2) + d + (b_1, b_2)$.

(4) If $a_1 > 0$ and $b_1 = 0$, similarly to (3), we have $0 < d < (a_1, a_2), (b_1, b_2)$ such that $-(a_1, a_2) + d + (a_1, a_2) = (0, t) = -(b_1, b_2) + d + (b_1, b_2)$. \square

Proposition 5.5. *A po-group G satisfies has wRDP if and only if, for each equation $u_1 + u_2 = v_1 + v_2$ of elements of G , there is a positive element $k \in G$ such that*

(P1) $v_2 \leq u_1 + k$;

(P2) $u_2 - k$ and $v_2 - k$ commute.

Proof. Let $u_1, u_2, v_1, v_2 \in G$ such that $u_1 + u_2 = v_1 + v_2$. Suppose (P1) and (P2) hold. We set $d_1 = v_2 - k$ and $d_2 = u_2 - k$. Clearly, $d_1 \leq v_2$ and $d_2 \leq u_2$. From (P1), it follows that $d_1 = v_2 - k \leq u_1$, and so $d_1 \leq u_1$. Also, $u_1 - (v_2 - k) + (u_2 - k) = u_1 + (u_2 - k) - (v_2 - k) = (u_1 + u_2) - k - (v_2 - k) = (v_1 + v_2) - k - (v_2 - k) = v_1$ and so $u_1 - d_1 = v_1 - d_2$ and $-u_1 + v_1 = -d_1 + d_2$. Since $0 \leq u_1 - d_1$, then $0 \leq v_1 - d_2$. That is, $d_2 \leq v_1$. Therefore, G satisfies RDP.

Conversely, let G satisfy have RDP. Set $k := -d_1 + v_2$. Then by (1) of Theorem 5.1(ii), $0 \leq k$ and $v_2 - k = d_1$. By (2) and (3), $u_2 - k = u_2 - v_2 + d_1 = -u_1 + v_1 + d_1 = d_2$ and so by (2), $u_2 - k$ and $v_2 - k$ commute. Therefore, (P1) and (P2) hold. \square

Proposition 5.6. *Let G be a po-group with RDP such that, for each equation $u_1 + u_2 = v_1 + v_2$ of elements of G , there is an element $k \in G^+$ satisfying (P1)–(P2) of Proposition 5.5, then we always can find a table*

u_1	c_{11}	c_{12}
u_2	c_{21}	c_{22}
	v_1	v_2

such that $c_{11}, c_{22} \in G^+$.

Proof. Let a po-group G with RDP satisfying (P1)–(P2) be given. According to Proposition 5.5 and Theorem 5.1(iii), $A \overrightarrow{\times} G$ has RDP for each po-group A with RDP, in particular, for the ℓ -group $A = \mathbb{R} \times \mathbb{R}$. Take positive elements $((1, 0), u_1), ((0, 1), u_2), ((0, 1), v_1), ((1, 0), v_2)$ from $(\mathbb{R} \times \mathbb{R}) \overrightarrow{\times} G$ such that $((1, 0), u_1) + ((0, 1), u_2) = ((0, 1), v_1) + ((1, 0), v_2)$. We can find an RDP table as follows

$((0, 1), v_1)$	(e_{11}, c_{11})	(e_{12}, c_{12})
$((1, 0), v_2)$	(e_{21}, c_{21})	(e_{22}, c_{22})
	$((1, 0), u_1)$	$((0, 1), u_2)$

Since $(0, 1) \wedge (1, 0) = (0, 0)$ in $\mathbb{R} \times \mathbb{R}$, we have $e_{11} = e_{22} = (0, 0)$. Therefore, $c_{11} \geq 0$ and $c_{22} \geq 0$. \square

Remark 5.7. (1) We note that there is an Abelian po-group with RDP that does not satisfy the conditions (P1)–(P2) of Proposition 5.5. Let $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x + y = 0\}$ be a po-subgroup of $\mathbb{R} \times \mathbb{R}$ ordered with respect to the original order in $\mathbb{R} \times \mathbb{R}$. Then $G^+ = \{(0, 0)\}$, so G is an Abelian po-group satisfying RDP. But G is not directed, so by (iii) of Theorem 5.1(iii) and Proposition 5.5, it does not satisfies (P1)–(P2).

(2) Similarly, every non-directed po-group does not satisfy (P1)–(P2).

(3) Every directed Abelian po-group, G , satisfies (P1)–(P2). Indeed, let $u_1 + u_2 = v_1 + v_2$ hold. Due to directness of G , there is an element $k \in G$ such that $k \geq 0, v_2 - u_1$. Then $v_2 \leq u_1 + k$ and of course, $u_2 - k$ and $v_2 - k$ commute.

Question. Does there exist a directed po-group with RDP and not with wRDP, or does every directed po-group with RDP satisfies wRDP?

A partial answer to this question is the following example.

Example 5.8. There is a directed po-group with RDP_0 and not satisfying RDP which does not satisfy wRDP.

Proof. We use an example from Remark 2.1. Thus let G be an additive group generated freely by the countably many elements g_0, g_1, \dots , let $v : (G; +, 0) \rightarrow (\mathbb{R}; +, 0)$ be the homomorphism determined by the conditions $v(g_{2i}) = v(g_{2i+1}) = (1/2)^i$, $i = 0, 1, \dots$. Define a partial order in G by setting $G^+ := \{x \in G \mid x = 0 \text{ or } v(x) > 0\}$. This means that we have for $a, b \in G$ $a \leq b$ iff $a = b$ or $v(a) < v(b)$. In [DvVe1, Ex 3.6], there was proved that G satisfies RDP_0 but RDP fails.

We assert that for this G , conditions (P1)–(P2) fail. Take elements g_1, g_2, g_3 and we put $u_1 = g_3 - g_1$, $u_2 = g_1$, $v_1 = g_3 - g_2$, and $v_2 = g_2$. Then $u_1 + u_2 = v_1 + v_2$. We show that there is no $k \geq 0$ such that $v_2 \leq u_1 + k$, and $u_2 - k$ and $v_2 - k$ commute. From construction of G , we see that k has to be strictly positive. Then we have to verify

$$g_2 \leq (g_3 - g_1) + k \quad \text{and} \quad g_1 - k + g_2 - k = g_2 - k + g_1 - k. \quad (5.1)$$

Assume that conditions (P1)–(P2) hold for this case. Then $u_2 - k$ and $v_2 - k$ commute, i.e. $g_2 - k + g_1 = g_1 - k + g_2$.

To prove a contradiction, we use the word techniques. The free group with generators g_0, g_1, \dots can be identified with the set of reduced words $n_1 g_{n_1} + \dots + n_l g_{n_l}$ for $n_1, \dots, n_l \in \{-1, 1\}$ over the alphabet $g_0, -g_0, g_1, -g_1, \dots$, where reduced means that there are no successive letters $g_i, -g_i$ or $-g_i, g_i$ in the word.

Take two words $g_2 - k + g_1$ and $g_1 - k + g_2$ which are identifiable with the same element and thus, in particular, of the same length. Hence either both are reduced or both are not reduced. Let $k = k_1 + \dots + k_m$ be the reduced word. Comparing the same words $g_2 - k_m - \dots - k_1 + g_1$ and $g_1 - k_m - \dots - k_1 + g_2$, we see that both words are not reduced, hence, $k = n(g_1 - g_2) + g_1$ for $n \geq 0$, or $k = n(g_2 - g_1) + g_2$ for $n \geq 0$. Since $v(g_3) = 1/4$ and $v(g_1) = v(g_2) = 1/2$, for $k = n(g_1 - g_2) + g_1$, we have $(g_3 - g_1) + k = (g_3 - g_1) + g_1 + n(-g_2 - g_1) = g_3 + n(-g_2 + g_1) < g_2$ while $1/3 = v(g_3 + n(-g_2 + g_1)) < 1/2 = v(g_2)$ which contradicts (5.1). Similarly, if $k = n(g_2 - g_1) + g_2$, we have $v(g_3 - g_1 + n(g_2 - g_1) + g_2) = v(g_3) < v(g_2)$, i.e. $g_3 < g_2$ which also contradicts (5.1).

Hence, (P1)–(P2) fail in G . □

The latter example can be generalized as follows.

Example 5.9. Let G be an additive group generated freely by the countably many elements g_0, g_1, \dots , let $v : (G; +, 0) \rightarrow (\mathbb{R}; +, 0)$ be the homomorphism determined by the conditions $v(g_i) > 0$, $i = 0, 1, \dots$ and $\liminf_n v(g_n) = 0$. Define a partial order in G by setting $G^+ := \{x \in G \mid x = 0 \text{ or } v(x) > 0\}$. Then G is a directed po-group with RDP_0 but RDP and sRDP fail in G .

Proof. The range of v , $v(G)$ is a subgroup of \mathbb{R} . Since $\liminf_n v(g_n) = 0$, we have that $v(G)$ is dense in \mathbb{R} , see [Go, Lem 4.21]. By [Fuc1, Ex. 10], G has RIP, which by Lemma 2.4 gets G has RDP_0 .

To exhibit RDP, take $a_1, a_2, b_1, b_2 \in G^+$ such that $a_1 + a_2 = b_1 + b_2$. Without loss of generality we can assume $a_1, a_2, b_1, b_2 > 0$. For it we search an RDP table in the form

$$\begin{array}{c|cc} a_1 & a_1 - k & k \\ a_2 & k - a_1 + b_1 & -k + b_2 \\ \hline & b_1 & b_2 \end{array},$$

where $k \geq 0$. Hence, $k - a_1 + b_1 = a_2 - b_2 + k$. If $v(a_1) < v(b_1)$, we put $k = 0$. If $v(a_1) > v(b_1)$, we put $k = -b_1 + a_1$. If $v(a_1) = v(b_1)$, the case $a_1 = b_1$ is trivial, so let $a_1 \neq b_1$. We claim that for such a case, there is no $k \geq 0$. Indeed, let a be any element of G such that $v(a) = 0$. Choose $b_1 = a_1 + a$, $b_2 = -a + a_2$. Then k has to commute with a , in particular, k has to commute with $a = g_1 + g_2 - g_1 - g_2$. Using word technique, we show that then $k = n(g_1 + g_2 - g_1 - g_2)$ for some $n \in \mathbb{Z}$. Indeed, let $k = k_1 + \dots + k_m$ be a reduced word and take two same words $k_1 + \dots + k_m + g_1 + g_2 - g_1 - g_2$ and $g_1 + g_2 - g_1 - g_2 + k_1 + \dots + k_m$. They are simultaneously reduced or non-reduced.

Let the words be reduced. It is possible to show that $m = 4j$. Comparing letters, we have $k_1 = g_1 = k_{m-3}$, $k_2 = g_2 = k_{m-2}$, $k_3 = -g_1 = k_{m-1}$, $k_4 = -g_2 = k_m$. Hence $k_5 + \dots + k_{m-4} + g_1 + g_2 - g_1 - g_2 = g_1 + g_2 - g_1 - g_2 + k_5 + \dots + k_{m-4}$, which gives after finitely many cases $k = n(g_1 + g_2 - g_1 - g_2)$ for some $n \geq 1$.

Let the words be not reduced, then $k_m = -g_1$ and $k_1 = g_2$, and similarly we can show that then $k = n(g_2 + g_1 - g_2 - g_1)$ for some $n \geq 0$.

Since $v(k) = nv(g_1 + g_2 - g_1 - g_2) = 0$, then $k \geq 0$ only if $k = 0$ which implies that $v(k - a_1 + b_1) = 0$ but $a_1 + b_1$ is not positive. Hence, RDP fails in G .

To prove that (P1)–(P2) fail, let $u_1 + u_2 = v_1 + v_2$ be given, where $u_2 = g_1 + g_2$, $v_2 = g_2 + g_1$, and $u_1 < 0$. Then also $v_1 < 0$. Using word technique, to find a solution for $u_2 - k + v_2 = v_2 - k + u_2$, we exhibit words $-g_1 - g_2 + k_1 + \dots + k_m - g_2 - g_1 = -g_2 - g_1 + k_1 + \dots + k_m - g_1 - g_2$. The equation has a solution only if $m = 4i + 2$, and then $k = n(g_1 + g_2 - g_1 - g_2) + g_1 + g_2$ or $k = n(g_2 + g_1 - g_2 - g_1) + g_2 + g_1$ for $n \geq 0$. Check $v(u_1 + k) = v(u_1) + v(n(g_1 + g_2 - g_1 - g_2)) + v(g_1 + g_2) < v(g_1 + g_2) = v(v_2)$ which entails $u_1 + k < v_2$. The same is true for the second solution of k . Hence, wRDP fails in G . \square

If in Example 5.9, we assume that G is an Abelian group freely generated by g_1, g_2, \dots and the order is the same as in Example 5.9, then G is a directed po-group with RDP and with wRDP.

6 Conclusion

Let \mathcal{LRDP} be the class of po-groups A with RDP such that $A \overrightarrow{\times} G$ has RDP for each directed po-group G with RDP. We have shown that the class \mathcal{LRDP} contains all

- (i) linearly ordered groups, [DvKo, Thm 3.1], Theorem 3.2;
- (ii) antilattice po-groups with RDP, Theorem 3.1;
- (iii) direct products of linearly ordered Abelian groups satisfying the condition of Theorem 3.2 and with the strict product ordering, Theorem 3.2;
- (iv) po-groups with RDP satisfying NCDP Theorem 5.1(i).

In the paper we have obtained other interesting conditions when the lexicographic product of two po-groups has RDP.

It is still an open problem whether e.g. every po-group with RDP belongs \mathcal{LRDP} or a weaker problem whether every Abelian po-group with RDP belongs to \mathcal{LRDP} .

The study of the lexicographic product of po-groups is important for the study of so-called lexicographic pseudo effect algebras, i.e. when we can represent a pseudo effect algebra as an interval in a unital po-group $(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is a unital po-group with RDP and G is a directed po-group with RDP. The authors hope to continue in these applications of the lexicographic product of po-groups for pseudo effect algebras.

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